



Friday 5:49 AM

Let M be Medvedev's logic. It is known by work of Zacharyashev that M , like every superintuitionistic logic, can be axiomatized by canonical formulas. M has many special properties: the disjunction property and structural completeness amongst the more non-trivial.

Please make the following precise against the literature: the formulas arising in this axiomatization, of the form $\rho(A, D)$ for A a subdirectly irreducible and D a domain, such that every Heyting algebra $H \Vdash \rho(A, D)$ if and only if the dual space of A embeds into the dual of H , via a closed embeddings preserving some elements from the domain.

Then check whether the structural hypotheses allow a simplification of the axiomatization by Zacharyashev.

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The precise result is a quotient-partial-embedding theorem. Dually, X_A is generally a

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partial p-morphic image of a closed upset of X_H , not a closed subspace of X_H . For Medvedev logic, structural completeness does remove the intermediate quotient from the global refutation test; the disjunction property then rules out all empty-domain canonical axioms.

1. The canonical formula

A notation warning first. Zakharyashev writes frame-based formulas $\chi(F, \mathcal{D}, \perp)$, later usually $\beta(F, \mathcal{D}, \perp)$. Their algebraic counterpart is denoted $\alpha(A, D, \perp)$ by Bezhanishvili–Bezhanishvili. I will write

$$\rho(A, D) := \alpha(A, D, \perp)$$

as in the question. In later work, however, $\rho(A, D)$ often denotes a different, multi-conclusion stable canonical rule.

Let A be a finite, nontrivial, subdirectly irreducible Heyting algebra, and let s_A be its opremum—the greatest element below 1. Let $D \subseteq A^2$. For every $a \in A$, introduce a variable p_a , and put

$$\begin{aligned} \delta_{A,D} := & \bigwedge_{a,b \in A} (p_{a \wedge b} \leftrightarrow (p_a \wedge p_b)) \\ & \wedge \bigwedge_{a,b \in A} (p_{a \rightarrow b} \leftrightarrow (p_a \rightarrow p_b)) \\ & \wedge \bigwedge_{a \in A} (p_{\neg a} \leftrightarrow \neg p_a) \\ & \wedge \bigwedge_{(a,b) \in D} (p_{a \vee b} \leftrightarrow (p_a \vee p_b)). \end{aligned}$$

Then

$$\boxed{\rho(A, D) := \delta_{A,D} \rightarrow p_{s_A}.$$

Thus D is a set of pairs, not a set of elements. The formula records the whole bounded implicative-semilattice reduct $(A, \wedge, \rightarrow, 0)$, but records joins only on the designated pairs in D .

2. Exact algebraic and dual refutation

theorem

Write $e : A \hookrightarrow_D C$ when e is an injective $(\wedge, \rightarrow, 0)$ -homomorphism and

$$e(a \vee b) = e(a) \vee e(b) \quad \text{for every } (a, b) \in$$

The Subframe Jankov Lemma says that for every Heyting algebra H ,

$$H \not\models \rho(A, D) \iff \exists q : H \twoheadrightarrow C \quad \exists e : (A) \hookrightarrow_D C$$

The embedding may preserve further joins accidentally; preservation is required at least on D . This is Definition 5.1 and Theorem 5.3 of Bezhanishvili–Bezhanishvili 2009 [arXiv:0905.0001](#), pp. 538–539, giving the algebraic form of Zakharyashev's original theorem [arXiv:0905.0001](#).

Let X_K denote the Esakia dual of K . The dual diagram of (1) is

$$X_A \xleftarrow[\text{onto}]{\text{well partial Esakia map } f} Y = X_C \xrightarrow{\text{closed-upset isomorphism}} X_D \quad (2)$$

Here “well” means that the domain of f is cofinal:

$$Y = \downarrow \text{dom}(f).$$

The map also satisfies the closed-domain condition associated with D . Explicitly, writing

$$\hat{a} = \{x \in X_A : a \in x\}, \quad f[\uparrow y] = f((\uparrow y) \cap \text{dom}(f))$$

the condition for $(a, b) \in D$ can be written

$$f[\uparrow y] \subseteq \hat{a} \cup \hat{b} \implies f[\uparrow y] \subseteq \hat{a} \text{ or } f[\uparrow y] \subseteq \hat{b} \quad (3)$$

for every $y \in Y$. This is exactly dual to preservation of $a \vee b$.

Consequently, the formulation " X_A embeds closedly into X_H " reverses the relevant arrow. The closed embedding is $Y \hookrightarrow X_H$, dual to the quotient $H \twoheadrightarrow C$; X_A is an onto partial Esakia image of Y . Corollary 5.5 of the same paper gives this formulation.

For every formula φ , selective filtration produces finitely many pairs (A_i, D_i) such that

$$\text{IPC} \vdash \varphi \leftrightarrow \bigwedge_i \rho(A_i, D_i).$$

The A_i are finite SI quotient images of the finite $(\wedge, \rightarrow, 0)$ -structure generated by the relevant subformula values, while D_i records the joins that survive that construction. This is Theorem 5.7 and Corollaries 5.10 and 5.13 of the 2009 paper.

3. Specialization to Medvedev logic

Let

$$\mathfrak{F}_n = (\wp(n) \setminus \{\emptyset\}, \supseteq), \quad B_n = \text{Up}(\mathfrak{F}_n).$$

Then

$$M = \bigcap_{n \geq 1} \text{Log}(B_n).$$

Hence the general canonical basis can be written

$$M = \text{IPC} + \Sigma_M,$$

where

$$\Sigma_M = \{\rho(A, D) : \forall n \ B_n \models \rho(A, D)\} .(3)$$

By (1),

$$M \not\models \rho(A, D) \iff \exists n, C \ (B_n \twoheadrightarrow C \ \& \ A \not\rightarrow C) .(4)$$

The definition of M and its DP and structural completeness are reviewed in Přenosil 2024 ↗ and Chen–Ding 2024 ↗.

4. What structural completeness genuinely simplifies

There is a stronger M -specific criterion:

$$\boxed{M \not\models \rho(A, D) \iff \exists n \ A \hookrightarrow_D B_n.} \quad (5)$$

Thus, globally over the Medvedev algebras, the quotient C in (4) can be removed. The target B_n obtained from structural completeness need not be the same B_n occurring in an original quotient witness.

Here is the short proof. Write $\rho(A, D) = \delta_{A,D} \rightarrow p_{s_A}$. If $M \not\models \rho(A, D)$, then the rule

$$\frac{\delta_{A,D}}{p_{s_A}}$$

is nonderivable. By Prucnal's structural completeness theorem it is nonadmissible, so there is a substitution σ such that

$$M \vdash \sigma(\delta_{A,D}), \quad M \not\models \sigma(p_{s_A}).$$

Choose n and an evaluation v in B_n with $v(\sigma(p_{s_A})) < 1$, and define

$$e(a) = v(\sigma(p_a)).$$

The diagram $\sigma(\delta_{A,D})$ shows that e preserves $\wedge, \rightarrow, 0$ and all joins in D . It is injective: if $a \neq b$ but $e(a) = e(b)$, then

$$c = (a \rightarrow b) \wedge (b \rightarrow a) < 1,$$

hence $c \leq s_A$, while $e(c) = 1$, forcing $1 \leq e(s_A)$, a contradiction. The converse follows by using an embedding e as the valuation $p_a \mapsto e(a)$.

Dually, (5) becomes

$$M \not\models \rho(A, D) \iff \exists n \ \mathfrak{F}_n \dashrightarrow X_A$$

via an onto well partial Esakia morphism satisfying CDC_D . The preliminary closed upset $Y \subseteq \mathfrak{F}_n$ disappears, although the resulting morphism remains partial.

A useful finite redundancy reduction follows. For each reduct embedding $e : A \hookrightarrow B_n$, put

$$\text{Def}(e) = \{(a, b) \in A^2 : e(a \vee b) \neq e(a) \vee e(b)\}$$

Then

$$M \vdash \rho(A, D) \iff D \cap \text{Def}(e) \neq \emptyset \text{ for all } e \in \text{Emb}(A, B_n)$$

So for each A , it suffices to retain the inclusion-minimal D 's that hit every join-defect set. This is a genuine semantic simplification, though not currently an effective enumeration because n is unbounded.

5. What the disjunction property rules out

If an intermediate logic has DP, its disjunction-free fragment is exactly that of IPC—

Zakharyashev 1994, Theorem 6 $\nearrow$, independently obtained by Minari.

Now $\rho(A, \emptyset)$ is disjunction-free and is refuted in A by the naming valuation $p_a \mapsto a$. Therefore

$$M \not\vdash \rho(A, \emptyset)$$

for every finite nontrivial SI A . So DP does not permit deleting the closed-domain data; it shows that the tempting empty-domain/cofinal-subframe axiomatization is impossible. Combined with (5), it also yields

$$\forall A \text{ finite SI} \quad \exists n \ A_{\wedge, \rightarrow, 0} \hookrightarrow (B_n)_{\wedge, \rightarrow, 0}.$$

At the opposite extreme, $D = A^2$ gives the ordinary Jankov formula $J(A)$. Replacing all partial domains by A^2 would amount to proving

that M is Jankov-axiomatizable, equivalently join-splitting. That is a separate condition; the relevant criterion is given in Bezhanishvili–Bezhanishvili, Theorems 3.9 and 3.13. Neither DP nor structural completeness implies it, and the literature cited above does not establish it for M .

There is a dramatic simplification at the rule level: Citkin observes that, because M has DP and is structurally complete, the single multi-conclusion rule

$$\frac{p \vee q}{p, q}$$

is a basis for all admissible multi-conclusion rules of M ; see Citkin, Remark 1 after Corollary 3.5. This concerns admissibility, not an axiomatization of the theorem set.

So the final assessment is:

- structural completeness removes the homomorphic-image/closed-upset stage from the global canonical refutation test for M ;
- DP eliminates every $D = \emptyset$ candidate and shows that nontrivial join-domain information is essential;
- neither property yields a reduction to ordinary Jankov formulas, a finite basis, or an effective canonical list. Indeed, M is not finitely axiomatizable, while its recursive axiomatizability/decidability remains open.

